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HARMONIC VIBRATIONS OF PRETWISTED PLATES

JØRGEN JUNCHER JENSEN

Department of Solid Mechanics, The Technical University of Denmark, Lyngby, Denmark

Abstract—In this paper shell theory is used to analyse the free vibrations of thin, uniformly pretwisted, rectangular plates, simply supported at two opposite edges and free at the other two edges.

From the shell equations we can deduce that coupling between torsional and bending vibrations never occurs independent of the pretwist. Furthermore, it is shown that the eigenfrequencies are in all cases even functions of the angle of pretwist.

When the pretwist is small the shell equations are solved by means of regular perturbation, and the solution shows that the ratios between the bending frequencies with and without pretwist can be either greater than or smaller than unity, depending on the geometry of the plate.

INTRODUCTION

THE development of turbine blades has resulted in a great deal of work on pretwisted beams and plates.

All the problems relating to pretwisted, rectangular beams have been solved using the ordinary Bernoulli–Euler beam theory. In particular, Troesch, Anliker and Ziegler [1] and Anliker and Troesch [2] have found the eigenfrequencies for all possible combinations of simple boundary conditions.

More difficulties arise when it is necessary to use Timoshenko's beam theory or, especially, when a shell theory is used, because of the complexity of the governing equations.

Most of the problems that have been solved by means of shell equations concern the static behaviour of thin, rectangular, pretwisted plates. Knowles and Reissner [3] determined the torsional rigidity and the axial stiffness as functions of the angle of pretwist, solving the differential equations by means of a perturbation method. The solution of the bending problem is given by Wan [4, 5] and by Maunder and Reissner [6].

In the determination of the eigenfrequencies a serious problem arises as even for a flat, rectangular plate, analytical solutions only exist for very few combinations of the simple boundary conditions. Nordgren [7] found the eigenfrequencies for pretwisted, rectangular plates with two opposite edges simply supported and the two other either free or simply supported. These solutions were based on a formulation of shallow shell solutions for elastokinetics given by Naghdi [8]. Furthermore, approximate formulae for the torsional frequencies are found by Reissner and Washizu [10] and by Di Prima [11].

The present work treats the same problem as [7], but a more general shell theory given by Niordson [9] will be used. The shell equations will be solved by a regular perturbation method when the pretwist is small.

1. SHELL THEORY

Surface geometry

As shown in Fig. 1, we choose a coordinate system so that a point (u^1, u^2) on the middle surface of the pretwisted plate has the cartesian coordinates

$$f^1 = u^1 \cos(u^2/k), \qquad f^2 = u^1 \sin(u^2/k), \qquad f^3 = u^2 \tag{1}$$

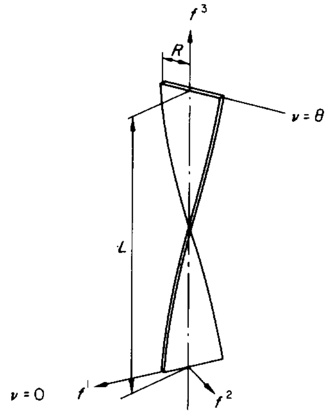


FIG. 1.

where

$$k = L/\Theta, \tag{2}$$

L is the length, R the half-width and Θ the total pretwist of the plate.

In the following we shall use, for the sake of brevity,

$$u^1 \equiv r \tag{3}$$

and in all definitions and relations to follow, Latin indices denote three-dimensional cartesian components, while Greek superscripts and subscripts refer to contravariant and covariant surface tensor components. The summation convention is applied as usual. Partial differentiation with respect to the surface coordinates is denoted by commas, and covariant differentiation (based on the geometry of the undeformed shell) is denoted D_α.

The metric tensor is

$$a_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \tag{4}$$

where

$$a = \det(a_{\alpha\beta}) = (r^2 + k^2)/k^2. \tag{5}$$

The normal to the plate

$$X^i = a^{-\frac{1}{2}}(\sin(u^2/k),\ -\cos(u^2/k),\ r/k). \tag{6}$$

The covariant and the mixed curvature tensor are

$$d_{\alpha\beta} = \frac{-1}{ka^{\frac{1}{2}}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

and

$$d^{\beta}_{\alpha} = \frac{-1}{ka^{\frac{1}{2}}}\begin{pmatrix} 0 & a^{-1} \\ 1 & 0 \end{pmatrix} \tag{8}$$

The only nonvanishing components of the Christoffel symbols are

$$\left\{ \begin{matrix} 1 \\ 2 \quad 2 \end{matrix} \right\} = \frac{-r}{k^2}, \qquad \left\{ \begin{matrix} 2 \\ 1 \quad 2 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 2 \quad 1 \end{matrix} \right\} = \frac{r}{ak^2}. \tag{9}$$

It should be noted that this geometry refers to the undeformed middle surface.

Deformations

According to the theory of shells given by Niordson [9], the deformations of the middle surface can be completely described by the membrane strain tensor

$$E_{\alpha\beta} = \tfrac{1}{2}(D_\alpha v_\beta + D_\beta v_\alpha) - d_{\alpha\beta} w \tag{10}$$

and the bending strain tensor

$$K_{\alpha\beta} = D_\alpha D_\beta w + d_{\alpha\gamma} D_\beta v^\gamma + d_{\beta\gamma} D_\alpha v^\gamma + v^\gamma D_\beta d_{\gamma\alpha} - d_{\beta\gamma} d^\gamma_\alpha w \tag{11}$$

where v^α and w are the displacements in the direction of the surface base vectors and in the direction of the surface normal, respectively. These displacements are functions of u^1, u^2 and time t. Niordson's shell equations are a consistent first-order, linear shell theory in the sense of Koiter [12].

The relations between v^α, w and the displacements $\bar{v}^i$ in the cartesian coordinate system are given by

$$\bar{v}^i = f^i_{,\alpha} v^\alpha + X^i w. \tag{12}$$

Hooke's law

For the homogeneous and elastically isotropic medium we assume the following constitutive equations

$$N^{\alpha\beta} = D_1((1-\nu)E^{\alpha\beta} + \nu a^{\alpha\beta} E^\gamma_\gamma) \tag{13}$$

and

$$M^{\alpha\beta} = D((1-\nu)K^{\alpha\beta} + \nu a^{\alpha\beta} K^\gamma_\gamma) \tag{14}$$

where E is Young's modulus, ν is Poisson's ratio, and h is the plate thickness. Furthermore, D and D_1 are defined by

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

and

$$D_1 = \frac{Eh}{1-\nu^2}. \tag{15}$$

The stresses and the couples are given in terms of the symmetric membrane stress tensor $N^{\alpha\beta}$ and the symmetric moment tensor $M^{\alpha\beta}$.

The equations of equilibrium

The equilibrium conditions are

$$D_\alpha N^{\alpha\beta} + 2d^\beta_\gamma D_\alpha M^{\alpha\gamma} + M^{\alpha\gamma} D_\alpha d^\beta_\gamma + F^\beta = 0 \tag{16}$$

and

$$D_\alpha D_\beta M^{\alpha\beta} - d_{\alpha\beta} d^\beta_\gamma M^{\alpha\gamma} - d_{\alpha\beta} N^{\alpha\beta} - p = 0 \tag{17}$$

where F^α and p are the external loads per unit area of the middle surface, acting in the directions of the surface base vectors and the surface normal, respectively. In the following, where we are concerned with free vibrations, the external loads are equal to the d'Alembert loads.

The d'Alembert loads per unit volume:

$$p^i = -\rho_m a^i \tag{18}$$

where ρ_m is the density and the acceleration a^i is determined by

$$a^i = \ddot{v}^i + \begin{Bmatrix} i \\ ks \end{Bmatrix} \dot{v}^k \dot{v}^s \simeq \ddot{v}^i. \tag{19}$$

We have neglected nonlinear terms because we are using a linear shell theory. A dot denotes differentiation with respect to time. In equation (19) v^3 means w.

According to [9] we transform the body forces p^i to equivalent surface forces as follows (where the variation of the displacements across the shell thickness is neglected):

$$F^\alpha = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} p^\gamma(\delta^\alpha_\gamma - d^\alpha_\gamma z)(1 - 2Hz + Kz^2)\,\mathrm{d}z = -\rho_m \ddot{v}^\alpha h\left(1 - \frac{1}{12}\frac{h^2}{k^2 a^2}\right) \tag{20}$$

and

$$p = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} p^3(1 - 2Hz + Kz^2)\,\mathrm{d}z = -\rho_m \ddot{w} h\left(1 - \frac{1}{12}\frac{h^2}{k^2 a^2}\right) \tag{21}$$

where we have made use of

$$H = \text{the mean curvature} = \tfrac{1}{2} d^\alpha_\alpha = 0$$

$$K = \text{the Gaussian curvature} = a^{-1}\det(d_{\alpha\beta}) = -(ka)^{-2}.$$

In the linear approximation, the free vibrations will be harmonic, i.e.

$$\ddot{v}^\alpha(u^1, u^2, t) = -\omega^2 v^\alpha(u^1, u^2, t) \quad \text{and} \quad \ddot{w}(u^1, u^2, t) = -\omega^2 w(u^1, u^2, t) \tag{22}$$

where ω are the eigenfrequencies of the shell.

From now on, v^α, w denote the displacements as functions of u^1, u^2 only, and the total displacements are then given by $v^\alpha \sin \omega t$ and $w \sin \omega t$, with suitable choice of the time origo. The same applies to all other functions of v^α and w, because of the linearity, and in the following these functions represent the (u^1, u^2) dependent parts only.

Boundary conditions

The plate under consideration is simply supported at $u^2 = 0, L$ and free at $u^1 = \pm R$. According to the theory of shells [9], the boundary conditions on a free edge are

$$T^\alpha = Q = M_B = 0 \tag{23}$$

and the boundary conditions at a simply supported edge, with normal constraint, are

$$w = v^\alpha n_\alpha = T^\alpha t_\alpha = M_B = 0 \tag{24}$$

where the effective boundary membrane force per unit length is

$$T^\alpha = (N^{\alpha\beta} + d^\alpha_\gamma M^{\beta\gamma} + d^\alpha_\gamma t_\sigma t^\gamma M^{\beta\sigma}) n_\beta, \tag{25}$$

the effective transverse force per unit length is

$$Q = -(D_\alpha M^{\alpha\beta}) n_\beta - \frac{\partial}{\partial s}(M^{\alpha\beta} n_\alpha t_\beta), \tag{26}$$

and the bending moment per unit length is

$$M_B = M^{\alpha\beta} n_\alpha n_\beta. \tag{27}$$

Furthermore, n_α denotes the unit normal vector to the edge (in direction outward from the edge), t_α denotes the unit tangent vector, and s measures length along the edge.

Between n_α and t_α there exists the following relation

$$n_\alpha = \varepsilon_{\alpha\beta} t^\beta \tag{28}$$

where

$$\varepsilon_{\alpha\beta} = a^{\frac{1}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{29}$$

Furthermore,

$$\frac{\partial}{\partial s} = \frac{\mathrm{d}u^\alpha}{\mathrm{d}s} \frac{\partial}{\partial u^\alpha} = t^\alpha \frac{\partial}{\partial u^\alpha}. \tag{30}$$

Application of (23–30) results in the following boundary conditions at the free edges where $n_\alpha = \pm(1, 0)$

$$\begin{aligned} &N^{11} - \frac{2}{ka^{\frac{1}{2}}} M^{21} = 0, \qquad N^{21} = 0, \\ &M^{11}_{,1} + 2M^{21}_{,2} - \frac{r}{k^2} M^{22} = 0, \qquad M^{11} = 0 \end{aligned} \tag{31}$$

and at the simply supported edges, where $n_\alpha = \pm a^{\frac{1}{2}}(0, 1)$,

$$\begin{aligned} &w = v^2 = 0, \\ &N^{21} = 0, \qquad M^{22} = 0. \end{aligned} \tag{32}$$

Now the complete set of equations to determine $v^\alpha = v^\alpha(u^1, u^2)$, $w = w(u^1, u^2)$ and ω are given.

2. NON-DIMENSIONALIZATION AND SIMPLIFICATION

We introduce the following dimensionless functions

$$
\begin{aligned}
\rho &\equiv r/R \\
\gamma &\equiv h/R \\
\lambda &\equiv R/k = R\Theta/L \\
w &\equiv w/R \\
v^{\alpha} &\equiv v^{\alpha}/R \\
m^{\alpha\beta} &\equiv M^{\alpha\beta}/RD_1 \\
n^{\alpha\beta} &\equiv N^{\alpha\beta}/D_1 \\
\Omega &\equiv \omega R/(D_1/\rho_m h)^{\frac{1}{2}} = \omega L^2/(D/\rho_m h)\cdot\frac{1}{\sqrt{12}}\left(\frac{R}{L}\right)^2\gamma.
\end{aligned}
\tag{33}
$$

The last equation shows the relation between Ω and $\omega L^2/(D/\rho_m h)^{\frac{1}{2}}$, which is normally used in the literature as dimensionless frequency.

Substitution of equation (33) in the complete set of equations transforms these into the following form (where $E^{\alpha\beta}$ and $K^{\alpha\beta}$ have been eliminated)

$$
\begin{aligned}
a &= 1+\rho^2\lambda^2 \\
n^{11} &= v^1_{,1}+\nu\left(v^2_{,2}+\frac{\rho\lambda^2}{a}v^1\right) \\
n^{21} &= \tfrac{1}{2}(1-\nu)\left(v^2_{,1}+\frac{1}{a}v^1_{,2}+\frac{2\lambda}{a^{\frac{1}{2}}}w\right) \\
n^{22} &= \frac{1}{a}\left(v^2_{,2}+\frac{\rho\lambda^2}{a}v^1+\nu v^1_{,1}\right) \\
m^{11} &= \frac{\gamma^2}{12}\left(w_{,11}-\frac{2\lambda}{a^{\frac{1}{2}}}v^2_{,1}-\frac{\lambda^2}{a^2}w+\frac{\nu}{a}\left(w_{,22}+\rho\lambda^2 w_{,1}-\frac{2\lambda}{a^{\frac{1}{2}}}v^1_{,2}-\frac{\lambda^2}{a}w\right)\right) \\
m^{21} &= \frac{\gamma^2}{12}\,\frac{1-\nu}{a}\left(w_{,21}-\frac{\rho\lambda^2}{a}w_{,2}-\frac{\lambda}{a^{\frac{1}{2}}}\left(v^2_{,2}+v^1_{,1}+\frac{\rho\lambda^2}{a}v^1\right)\right) \\
m^{22} &= \frac{\gamma^2}{12}\,\frac{1}{a^2}\left(w_{,22}+\rho\lambda^2 w_{,1}-\frac{2\lambda}{a^2}v^1_{,2}-\frac{\lambda^2}{a}w+\nu a\left(w_{,11}-\frac{2\lambda}{a^{\frac{1}{2}}}v^2_{,1}-\frac{\lambda^2}{a^2}w\right)\right)
\end{aligned}
\tag{34}
$$

$$
n^{11}_{,1}+n^{21}_{,2}+\frac{\rho\lambda^2}{a}n^{11}-\rho\lambda^2 n^{22}-\frac{2\lambda}{a^{\frac{1}{2}}}\left(m^{21}_{,1}+m^{22}_{,2}+\frac{\rho\lambda^2}{a}m^{21}\right)+\Omega^2 v^1\left(1-\frac{1}{12}\,\frac{\lambda^2\gamma^2}{a^2}\right)=0
$$

$$
n^{21}_{,1}+n^{22}_{,2}+\frac{3\rho\lambda^2}{a}n^{21}-\frac{2\lambda}{a^{\frac{1}{2}}}(m^{11}_{,1}+m^{21}_{,2})+\Omega^2 v^2\left(1-\frac{1}{12}\,\frac{\lambda^2\gamma^2}{a^2}\right)=0
$$

$$m^{11}_{,11}+2m^{21}_{,21}+m^{22}_{,2}+\frac{2\rho\lambda^2}{a}m^{11}_{,1}+\frac{4\rho\lambda^2}{a}m^{21}_{,2}-\rho\lambda^2 m^{22}_{,1}-2\lambda^2 m^{22}$$

$$+\frac{2\lambda}{a^{\frac{1}{2}}}n^{21}-\Omega^2 w\left(1-\frac{1}{12}\frac{\lambda^2\gamma^2}{a^2}\right)=0$$

$$\rho=\pm 1: n^{11}-\frac{2\lambda}{a^{\frac{1}{2}}}m^{21}=n^{21}=m^{11}_{,1}+2m^{21}_{,2}-\rho\lambda^2 m^{22}=m^{11}=0$$

$$u^2/R=0,\ L/R: w=v^2=m^{22}=n^{21}=0.$$

Now $_{,1}$ and $_{,2}$ means differentiation with respect to ρ and u^2/R, respectively. It can be shown that these equations are in agreement with [3–5].

From equations (34), we can deduce the following results:

(i): The equations can be separated into a system of ordinary differential equations by the substitution

$$\begin{aligned} w(\rho,u^2/R,\lambda) &= A(\rho,\lambda)\sin(\eta u^2/R)\\ v^1(\rho,u^2/R,\lambda) &= B(\rho,\lambda)\cos(\eta u^2/R)\\ v^2(\rho,u^2/R,\lambda) &= C(\rho,\lambda)\sin(\eta u^2/R)\end{aligned} \tag{35}$$

where

$$\eta = m\pi\frac{R}{L}, \qquad m=1,2,\ldots. \tag{36}$$

In particular, it will be seen that the boundary conditions at $u^2/R=0$, L/R are identically satisfied.

Equations (35) show that the u^2-dependence of the displacements is independent of the pretwist λ.

(ii): Inspection of equation (34) shows that the solution depends on λ in one of the following ways

(a) $\begin{cases} w, m^{11}, m^{21}, m^{22}, \Omega & \text{even functions of } \lambda\\ v^1, v^2, n^{11}, n^{21}, n^{22} & \text{odd functions of } \lambda\end{cases}$

(b) $\begin{cases} v^1, v^2, n^{11}, n^{21}, n^{22}, \Omega & \text{even functions of } \lambda\\ w, m^{11}, m^{21}, m^{22} & \text{odd functions of } \lambda.\end{cases}$

When $\lambda=0$, i.e. without pretwist, the first solution is reduced to the normal plate-solution, where $v^1\equiv v^2\equiv n^{\alpha\beta}\equiv 0$, and the second solution is reduced to the disc-solution, where $w\equiv m^{\alpha\beta}\equiv 0$.

This shows, as is known, that there is no coupling between plate-like and disc-like vibrations for a flat plate. On the contrary, equations (34) show that such coupling occurs when the plate is pretwisted.

Note that the eigenfrequencies Ω are always even functions of λ, which, physically, means that they are independent of the direction of the pretwist.

(iii): We further see that the solution to equation (34) depends on ρ in one of the following two ways:

$$\begin{cases} w, v^1, m^{11}, m^{22}, n^{21} & \text{even functions of } \rho \\ v^2, m^{21}, n^{11}, n^{22} & \text{odd functions of } \rho \end{cases}$$

or

$$\begin{cases} v^2, m^{21}, n^{11}, n^{22} & \text{even functions of } \rho \\ w, v^1, m^{11}, m^{22}, n^{21} & \text{odd functions of } \rho. \end{cases}$$

This is due to the fact that a function, defined in a closed interval, can always be decomposed into an even and an odd part and that when an even/odd function is differentiated, we get an odd/even function.

The physical significance of these results is most easily seen by looking at the displacements, transformed to the more convenient cylindrical coordinate system,

$$\bar{v}_r = \text{the radial displacement} = v^1$$

$$\bar{v}_\Theta = \text{the circumferential displacement} = v^2 \frac{r}{k} - \frac{w}{a^{\frac{1}{2}}}$$

$$\bar{v}_a = \text{the axial displacement} = v^2 + \frac{r}{ka^{\frac{1}{2}}} w.$$

Thus the circumferential displacement is either an even or odd function of ρ and consequently, the bending and the torsional vibrations are always uncoupled.

Perturbation solution

When equation (35) is substituted in (34) we obtain a set of ordinary differential equations nonlinear in λ. These equations cannot be solved in analytic form for any value of λ, but in the following, they are solved on the assumption that the pretwist is small, i.e.

$$\lambda \ll 1. \tag{37}$$

To solve the equations (34) with (37), we expand all terms in equations (34) in power series in λ and obtain a new set of equations that are independent of λ. As unknown we have the coefficient functions in the power series,

$$\begin{aligned} A(\rho, \lambda) &= A_0(\rho) + \lambda A_1(\rho) + \lambda^2 A_2(\rho) + O(\lambda^3) \\ B(\rho, \lambda) &= B_0(\rho) + \lambda B_1(\rho) + \lambda^2 B_2(\rho) + O(\lambda^3) \\ C(\rho, \lambda) &= C_0(\rho) + \lambda C_1(\rho) + \lambda^2 C_2(\rho) + O(\lambda^3) \\ \Omega(\lambda) &= \Omega_0 + \lambda^2 \Omega_2 + O(\lambda^4). \end{aligned} \tag{38}$$

The coefficients with index O refer to the solution for a flat plate.

As a further simplification we will only consider one of the four possibilities mentioned before, namely plate-like bending vibrations, and consequently we find

$$A_{2i+1}(\rho) \equiv B_{2i}(\rho) \equiv C_{2i}(\rho) \equiv 0, \quad i = 0, 1, 2, \ldots \tag{39}$$

and

$$A_{2i}(\rho) = A_{2i}(-\rho)$$
$$B_{2i+1}(\rho) = B_{2i+1}(-\rho) \qquad (40)$$
$$C_{2i+1}(\rho) = -C_{2i+1}(-\rho).$$

The remaining coefficients, A_0, A_2, B_1, C_1, Ω_0 and Ω_2, which determine the solution with an error of $O(\lambda^3)$, are obtained from equation (34) with application of equations (35) and (37–40):

$$ODE:\quad A_0'''' - 2\eta^2 A_0'' + \left(\eta^4 - \frac{12}{\gamma^2}\Omega_0^2\right) A_0 = 0 \qquad (41)$$

$$BC:\quad [A_0'' - \nu\eta^2 A_0]_{\rho=\pm 1} = [A_0''' - (2-\nu)\eta^2 A_0']_{\rho=\pm 1} = 0, \qquad A_0(\rho) = A_0(-\rho)$$

$$ODE:\quad B_1'' + \tfrac{1}{2}(1+\nu)\eta C_1' - \tfrac{1}{2}(1-\nu)\eta^2 B_1 + \Omega_0^2 B_1 = -(1-\nu)\eta A_0 - \frac{\gamma^2}{6}(\eta^3 A_0 - \eta A_0'')$$

and

$$-\tfrac{1}{2}(1+\nu)\eta B_1' - \eta^2 C_1 + \tfrac{1}{2}(1-\nu)C_1'' + \Omega_0^2 C_1 = -(1-\nu)A_0' - \frac{\gamma^2}{6}(\eta^2 A_0' - A_0''') \qquad (42)$$

$$BC:\quad [C_1' - \eta B_1 + 2A_0]_{\rho=\pm 1} = \left[B_1' + \nu\eta C_1 - \frac{\gamma^2}{6}(1-\nu)\eta A_0'\right]_{\rho=\pm 1} = 0$$

$$ODE:\quad A_2'''' - 2\eta^2 A_2'' + \left(\eta^4 - \frac{12}{\gamma^2}\Omega_0^2\right) A_2 =$$
$$-\Bigg[2\rho A_0''' + (2\eta^2\rho^2 - (1+\nu))A_0'' + 2\rho\eta^2 A_0' + \Bigg((5+\nu-2\rho^2\eta^2)\eta^2 + \Omega_0^2$$
$$+ 2(1-\nu)\frac{12}{\gamma^2} - 2\Omega_0\Omega_2\frac{12}{\gamma^2}\Bigg)A_0\Bigg] - \Bigg[2\eta B_1'' - \left(2\eta^3 + (1-\nu)\eta\frac{12}{\gamma^2}\right)B_1$$
$$-2C_1''' + \left(2\eta^2 + (1-\nu)\frac{12}{\gamma^2}\right)C_1'\Bigg] \qquad (43)$$

$$BC:\quad [A_2'' - \nu\eta^2 A_2 - (-\nu\rho A_0' - (3-\nu+\nu\eta^2\rho^2)A_0 + 2\eta(1-\nu)B_1)]_{\rho=\pm 1} = 0$$

and

$$\Bigg[A_2''' - (2-\nu)\eta^2 A_2' - \Bigg(2C_1'' - 2(1-2\nu)\eta^2 C_1 - \Bigg((2-\nu)\rho^2\eta^2$$
$$+ \frac{\gamma^2}{3}(1-\nu)\eta^2 - 1\Bigg)A_0' - 3\eta^2\rho A_0\Bigg)\Bigg]_{\rho=\pm 1} = 0$$

where $'$ denotes differentiation with respect to ρ.

A_0 and Ω_0 are obtained from the eigenvalue problem (41), B_1 and C_1 from the boundary value problem (42), and A_2, Ω_2 from the boundary value problem (43) (see Appendix).

As the right-hand side of the differential equation (43) is of order (A_0/γ^2) the solution A_2 will be of the same order. Furthermore it can be seen using equations (34) and (38) that

$$A_{2i} = O(A_0/\gamma^{2i}) \qquad i = 1, 2, \ldots.$$

Then, to assure the validity of the perturbation solution, it is necessary that

$$\lambda \ll \gamma. \tag{44}$$

This inequality is a stronger limitation on λ than the assumption (37).

Error estimate

The shell theory given in [9] is afflicted with an error $O(\varepsilon)$, where

$$\varepsilon = \frac{h^2}{1^2} + \frac{h}{r_{\min}}.$$

Here 1 is a characteristic wavelength of the deformation pattern on the middle surface and $r_{\min}$ is the numerically smallest radius of curvature.

For the pretwisted plate it can be shown that

$$r_{\min} = ka > k,$$

and therefore, when we neglect terms of $O(\lambda^3)$ in the perturbation solution and make use of equation (44), our final solution in the (m, n)-mode has a total error $O(\varepsilon)$ where

$$\varepsilon = \gamma^2\left(n^2 + \left(\frac{R}{L}\right)^2 m^2\right) \tag{45}$$

m and n are the number of waves in the u^2-direction and in the u^1-direction, respectively.

Shallow shell solution

When we neglect all terms of order $O(\gamma^2)$ in the differential equations (34), it can be shown that they are reduced to the equations given by Nordgren [7], who based his investigation on a shallow shell theory given by Naghdi [8]. In the following, we will compare the results obtained from the two shell theories. From the reduced form of equation (43), it can be shown that

$$A_2 \sim \gamma^{-2}, \qquad \Omega_2 \sim \Omega_0^{-1} \sim \gamma^{-1}. \tag{46}$$

3. RESULTS

When equations (41–43) are solved analytically—a simple but laborious task (see Appendix)—we get the displacements and the eigenfrequencies.

For testing purpose, the equations were solved for the torsional vibrations and total agreement was found between the present solutions and the results given by Nordgren (Table 1 and Fig. 1 in [7]).

Bending eigenfrequencies

The eigenfrequencies ω are given in the form

$$\frac{\omega L^2}{\sqrt{(D/\rho_m h)}} = \frac{\omega_0 L^2}{\sqrt{(D/\rho_m h)}}\left(1 + \Theta^2 \frac{\Delta}{\gamma^2} + O(\varepsilon)\right) \tag{47}$$

where ω_0 are the corresponding eigenfrequencies for a flat plate given by Ω_0 and

$$\Delta = \Delta\left(\frac{R}{L}, \nu, \gamma\right) \equiv \frac{\Omega_2}{\Omega_0}\left(\frac{R}{L}\right)^2 \gamma^2. \tag{48}$$

The frequency functions Δ and $(\omega_0 L^2)/\sqrt{(D/\rho_m h)}$ are plotted in Fig. 2(a)–(d) as regards the four lowest bending eigenfrequencies. It should be remarked that according to (46), Δ is independent of γ in the shallow shell approximation, in contrast to the results obtained from the more general shell theory [9]. Results derived by the shallow shell theory are marked by "$\gamma \to 0$" in Fig. 2.

The parameter R/L, γ, n and m are chosen in such a way that the error $O(\varepsilon)$, where ε is given by equation (45), is much smaller than unity. From the frequency curves we can deduce the following properties of the natural frequencies corresponding to bending modes:

(i): For modes (m, n), $n \geq 2$ there is only a slight difference between the solutions obtained from the two shell theories, whereas for modes $(m, 1)$ there is significant disagreement.

(ii): The results derived from both shell theories show that for a pretwisted plate, we obtain the greatest increase in the bending eigenfrequencies compared with a flat plate when the dimension is nearly quadratic. For $R/L \to 0$ or $R/L \to \infty$, the shallow shell

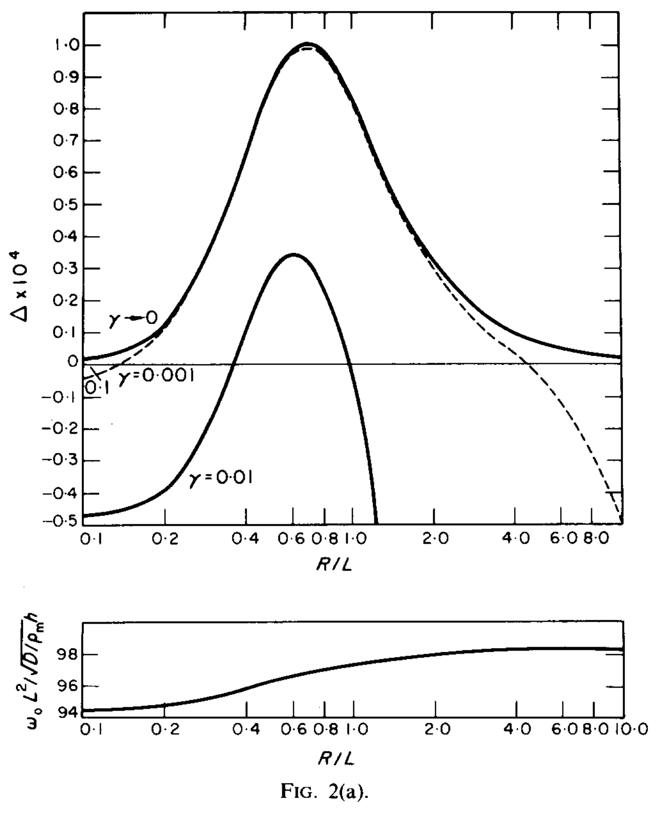
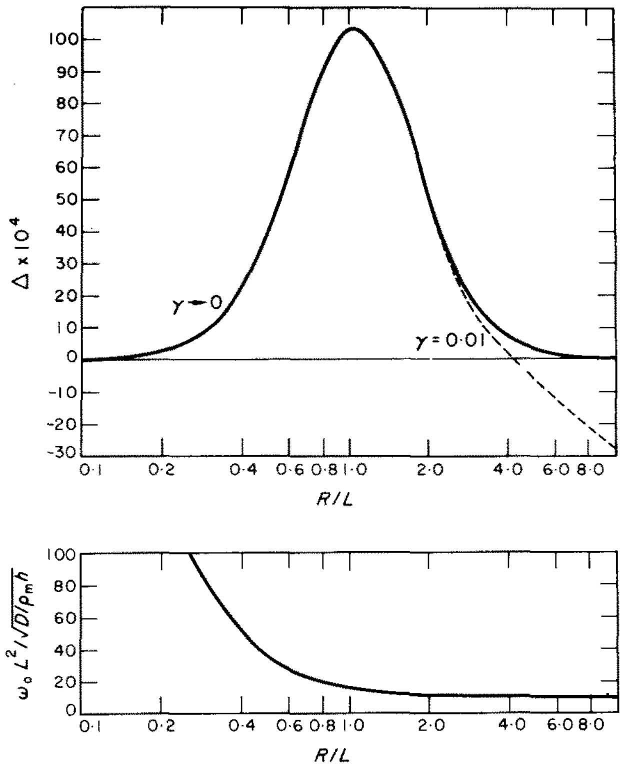
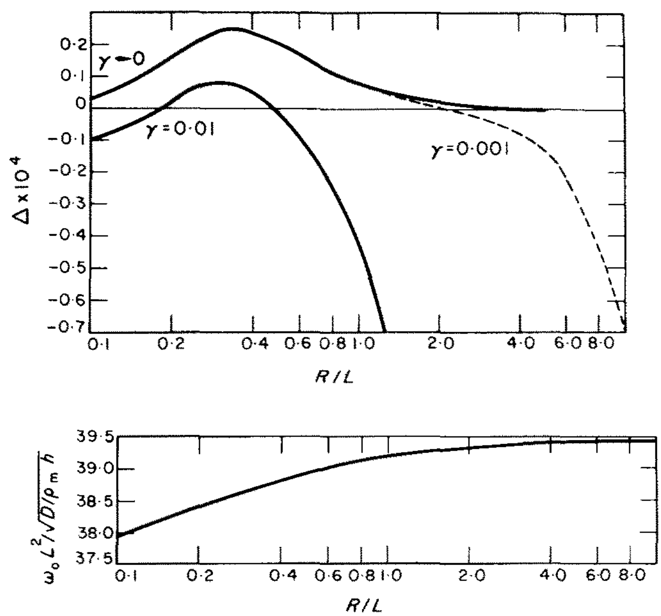


FIG. 2(a).



100
90
80
70
60
50
40
30
20
10
0
-10
-20
-30
Δ x 10⁴
γ→0
γ=0·01
0·1
0·2
0·4
0·6 0·8 1·0
2·0
4·0
6·0 8·0
R/L
100
80
60
40
20
0
ω₀L²/√(D/ρₘh)
0·1
0·2
0·4
0·6 0·8 1·0
2·0
4·0
6·0 8·0
R/L

FIG. 2(b).



0·2
0·1
0
-0·1
-0·2
-0·3
-0·4
-0·5
-0·6
-0·7
Δ x 10⁴
γ→0
γ=0·01
γ=0·001
0·1
0·2
0·4
0·6 0·8 1·0
2·0
4·0
6·0 8·0
R/L
39·5
39·0
38·5
38·0
37·5
ω₀L²/√(D/ρₘh)
0·1
0·2
0·4
0·6 0·8 1·0
2·0
4·0
6·0 8·0
R/L

FIG. 2(c).

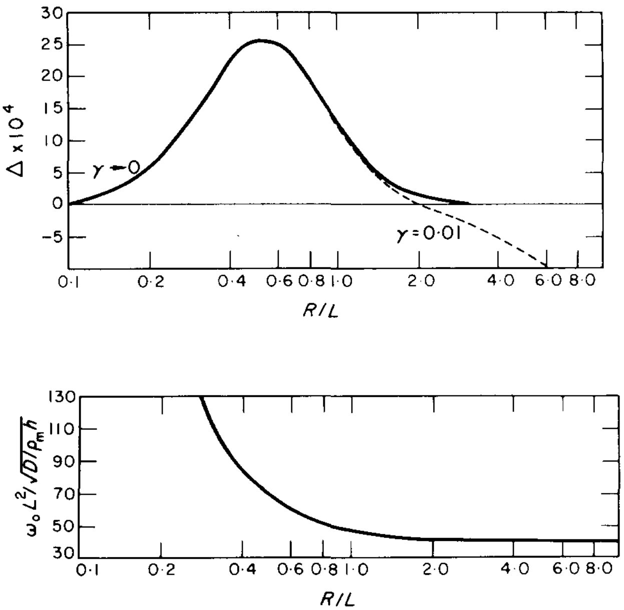


FIG. 2(d).

theory shows that the eigenfrequencies tend towards the flat plate solution. Niordson's shell theory, on the other hand, shows that the eigenfrequencies drop to a value below the eigenfrequencies of a flat plate.

(iii): The maximum increase in the eigenfrequencies is much smaller (with a factor 100) for modes (m, 1) than for other modes, and furthermore, this maximum value decreases for higher axial modes (m).

A reasonable explanation of the discrepancies mentioned in (i) and (ii) above is that Niordson's first-order shell equations take into account the tangential sliding of the plate along the simply supported edges. This effect is neglected in the shallow shell solution. The average tangential displacement amplitudes at the supports are particularly large in mode (m, 1). The inertia involved in the sliding will lower the eigenfrequencies and eventually neutralize, or even dominate, the effect of the increased bending stiffness gained by the pretwist.

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APPENDIX

When equation (41) is solved in analytic form, we obtain the eigenfrequency Ω_0 for mode (m, n) as the nth solution of

$$\frac{\alpha_-}{\alpha_+}\left(\frac{\alpha_+^2 - \nu\eta^2}{\alpha_-^2 - \nu\eta^2}\right)^2 - \frac{\tanh(\alpha_+)}{\tanh(\alpha_-)} = 0 \tag{A.1}$$

where

$$\eta = m\pi\frac{R}{L}$$

$$\alpha_+ = \left(\eta^2 + \frac{2\sqrt{3}}{\gamma}\Omega_0\right)^{\frac{1}{2}}$$

$$\alpha_- = \left(\eta^2 - \frac{2\sqrt{3}}{\gamma}\Omega_0\right)^{\frac{1}{2}}. \tag{A.2}$$

The corresponding displacement function

$$A_0(\rho) = K_0\{(\alpha_-^2 - \nu\eta^2)\cosh(\alpha_-)\cosh(\alpha_+\rho) - (\alpha_+^2 - \nu\eta^2)\cosh(\alpha_+)\cosh(\alpha_-\rho)\} \tag{A.3}$$

where K_0 is an arbitrary constant.

Equation (A.1) is solved numerically.

The solutions of the boundary value problem (42) are

$$B_1(\rho) = K_0\{b_1\cosh(\alpha\rho) + b_2\cosh(\beta\rho) + b_3\cosh(\alpha_+\rho) + b_4\cosh(\alpha_-\rho)\} \tag{A.4}$$

and

$$C_1(\rho) = K_0\{c_1\sinh(\alpha\rho) + c_2\sinh(\beta\rho) + c_3\sinh(\alpha_+\rho) + c_4\sinh(\alpha_-\rho)\} \tag{A.5}$$

where

$$\alpha = (\eta^2 - \Omega_0^2)^{\frac{1}{2}}$$

$$\beta = \left(\eta^2 - \frac{2}{1-\nu}\Omega_0^2\right)^{\frac{1}{2}} \tag{A.6}$$

and the coefficients b_i, c_i, $i = 1, 2, 3, 4$ are known functions of η, γ and ν, but independent of ρ.

The homogeneous part of the boundary value problem (43) is identical to the eigenvalue problem (41), hence, the inhomogeneous boundary conditions in (43) can only be satisfied for a single value of Ω_2, which is thereby determined.

The associated displacement function is

$$A_2(\rho) = K_0\{a_1\rho\sinh(\alpha_+\rho)+a_2\rho^2\cosh(\alpha_+\rho)+a_3\rho^3\sinh(\alpha_+\rho)+a_4\rho\sinh(\alpha_-\rho) \\ +a_5\rho^2\cosh(\alpha_-\rho)+a_6\rho^3\sinh(\alpha_-\rho)+a_7\cosh(\alpha\rho)+a_8\cosh(\beta\rho) \\ +a_9\cosh(\alpha_-\rho)\}+K_2\,.\,A_0(\rho) \tag{A.7}$$

where a_i, $i = 1, 2, \ldots, 9$ are known functions of η, γ and ν, and K_2 is another arbitrary constant.

To obtain single-valued displacements, it is necessary to normalize the displacements in some manner. Here we choose the condition

$$\int_{-1}^{1} |A(\rho, \lambda)|^2\,\mathrm{d}\rho \equiv 1. \tag{A.8}$$

With

$$A(\rho, \lambda) = A_0(\rho)+\lambda^2 A_2(\rho)+O(\lambda^4) \tag{A.9}$$

equation (A.8) becomes

$$\int_{-1}^{1} |A_0(\rho)|^2\,\mathrm{d}\rho = 1 \tag{A.10}$$

and

$$\int_{-1}^{1} A_0(\rho)A_2(\rho)\,\mathrm{d}\rho = 0. \tag{A.11}$$

Equations (A.10) and (A.11) determine the two arbitrary constants K_0 and K_2.

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Абстракт—В работе используется теория оболочек для анализа свободных колебаний тонких, равномерно предварительно скрученных, прямоугольных пластинок, шарнирно опертых на двух противоположенных краях и свободных на других.

Исходя из уравнений оболочек, можно проследить, что соединение между крутильными и изгибными колебаниями никогда происходит независимо от предварительного скручения. Кроме того, показывается что собственные частоты во всех случаях являются, даже, функциями угла предварительного скручения.

Для случая малого предварительного скручения уравнения оболочек решаюися способом нормальнього возмущения. В зависимости от геометрии пкастинки, решение указывает, что соотношения между частотами для изгиба с предварительным скручением или без могут достигать значения как более единицы, так и меньше.